Griffith University

3515ICT Theory of Computation

Kleene's Theorem

(Based loosely on slides by Harald Søndergaard of The University of Melbourne)

Kleene's Theorem

Theorem For every language L (over a finite alphabet Σ), the following statements are equivalent:

1. L is defined by some regular expression E.
2. L is accepted by some nondeterministic finite automaton N.
3. L is accepted by some deterministic finite automaton D.

NFAs $\Rightarrow$ DFAs

Lemma For every NFA N, there exists a DFA D s.t. $L(D) = L(N)$.

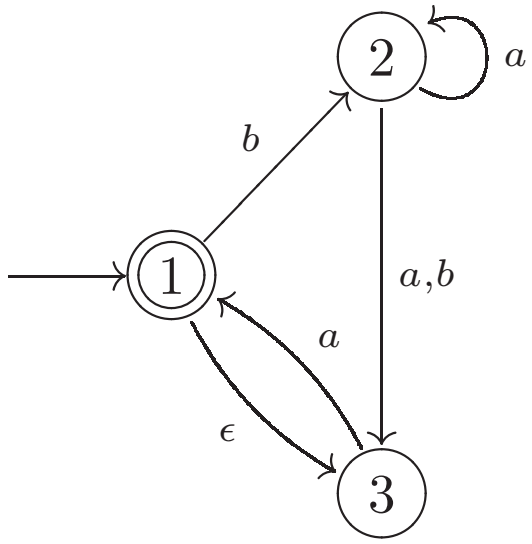
The proof is based on the so-called **subset** construction (Theorem 1.39, pp.55–58).

Given an NFA N, we construct a DFA D, each of whose states is a **set** of N-states.

If N has k states then D may have up to 2^k states (but it will often have far fewer than that).

Example 1.41, pp.57–58

Consider the NFA



We can systematically construct an equivalent DFA.

Its start state is $\{1, 3\}$.

From this state an a will take us back to $\{1, 3\}$.

From $\{1, 3\}$, b can only take us to $\{2\}$.

Continuing similarly gives the DFA.

Any state S which contains an accept state from the NFA will be an accept state for the DFA.

More formally...

Let $N = (Q, \Sigma, \delta, q_0, F)$.

Let $E(S)$ be the "ϵ-closure" of $S \subseteq Q$, *i.e.*, S together with all states reachable from S using only ϵ-transitions:

$$E(S) = \bigcup_{s \in S} \{\, s' \in Q \mid s \xrightarrow{*}_{\epsilon} s' \,\}$$

We construct $D = (Q', \Sigma, \delta', q_0', F')$ as follows.

- $Q' = 2^Q$
- $q_0' = E(\{q_0\})$.
- $\delta'(S, a) = \bigcup_{s \in S} E(\delta(s, a))$.
- $F' = \{\, S \in Q' \mid S \cap F \neq \emptyset \,\}$.

Note This construction may include some unreachable states.

Example 1.41 (cont.)

NFA:

	a	b	ϵ
$\rightarrow$ *1		2	3
2	23	3	
3	1		

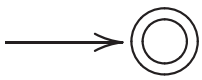
DFA:

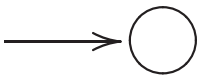
	a	b
$\rightarrow$ *13	13	2
2	23	3
23	123	3
3	13	$\emptyset$
*123	123	23
$\emptyset$	$\emptyset$	$\emptyset$

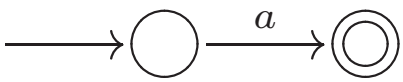
REs $\Rightarrow$ NFAs

Lemma For every regular expression E, there exists an NFA N s.t. $L(N) = L(E)$.

The proof is by **structural induction** on the form of E (Closure under regular operations, pp.58–63, and Lemma 1.55, pp. 67–69)

Case $E = \epsilon$: 

Case $E = \emptyset$: 

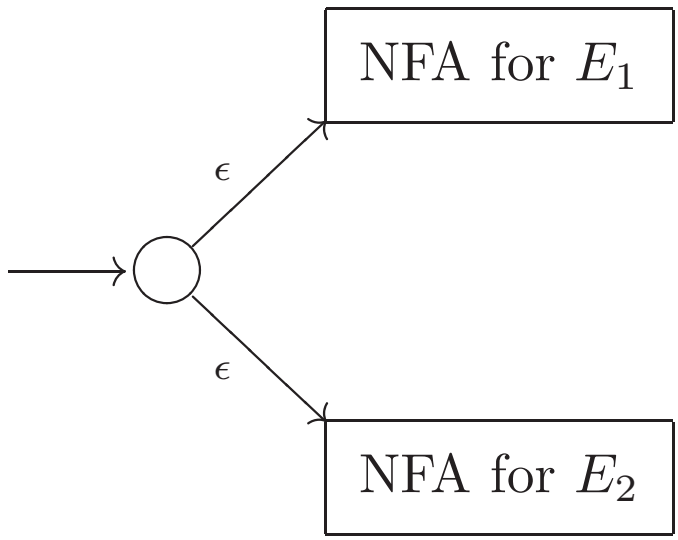
Case $E = a$: 

Cases $E = E_1 \cup E_2$, $E = E_1 E_2$, or $E = E_1^*$:
See following slides.

REs $\Rightarrow$ NFAs (cont.)

Case $E = E_1 + E_2$ (union):

By induction, suppose there are NFAs that accept $L(E_1)$ and $L(E_2)$. Then the following NFA accepts $L(E_1 + E_2)$.

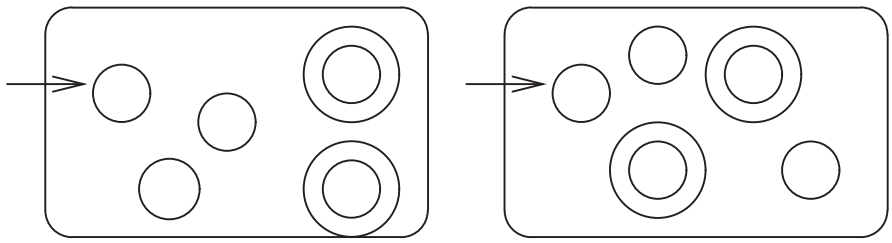


Here $F_E = F_1 \cup F_2$.

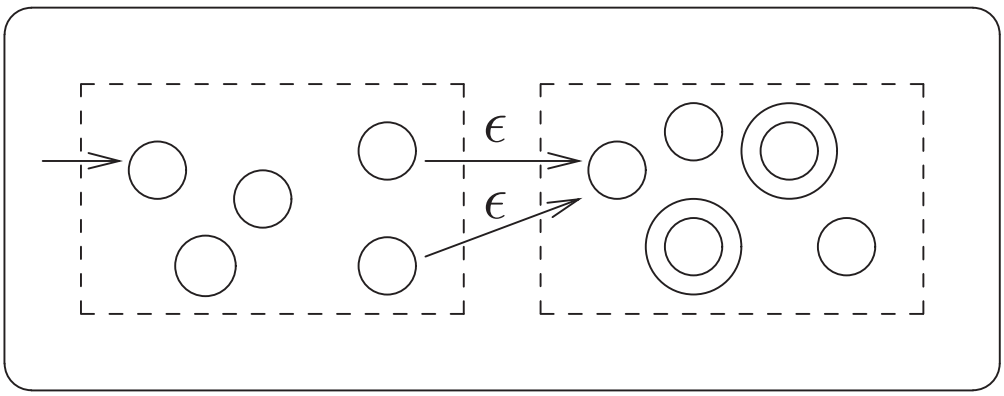
REs $\Rightarrow$ NFAs (cont.)

Case $E = E_1 E_2$ (concatenation):

By induction, suppose there are NFAs that accept $L(E_1)$ and $L(E_2)$



Then the following NFA accepts $L(E_1 E_2)$:

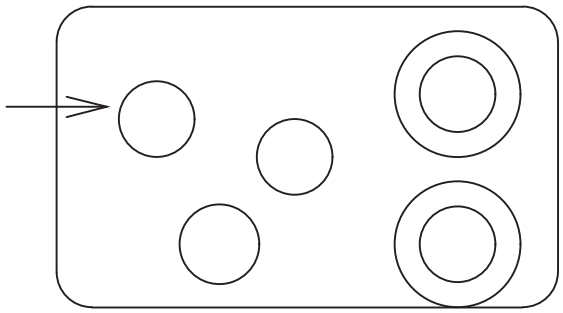


Here $F_E = F_2$.

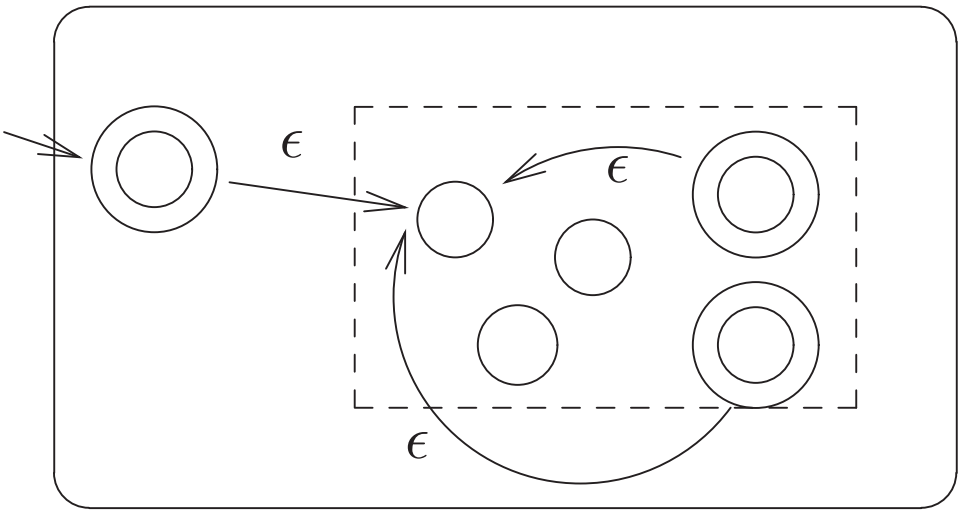
REs $\Rightarrow$ NFAs (cont.)

Case $E = E_1^*$ (repetition):

By induction, suppose there is an NFA that accepts $L(E_1)$.



Then the following NFA accepts $L(E_1^*)$:

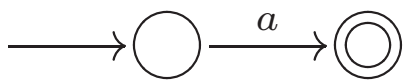


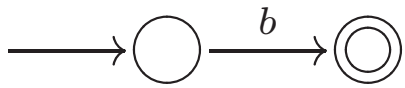
Here, $F_E = \{q_0\} \cup F_1$.

Example

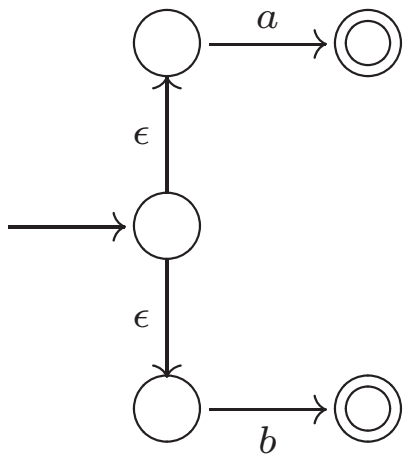
Let us construct an NFA for $(a \cup b)^* bc$

Start from innermost expressions and work out:

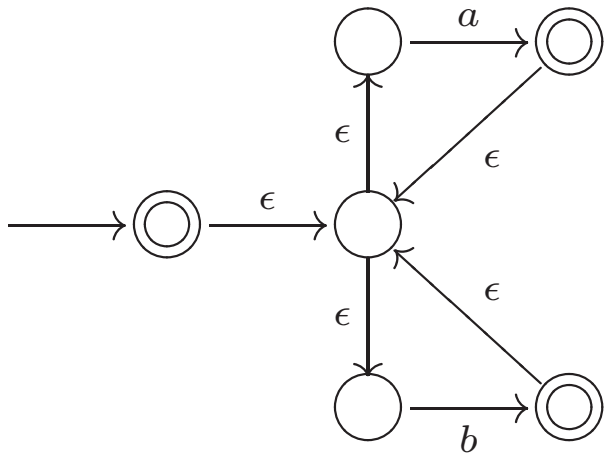




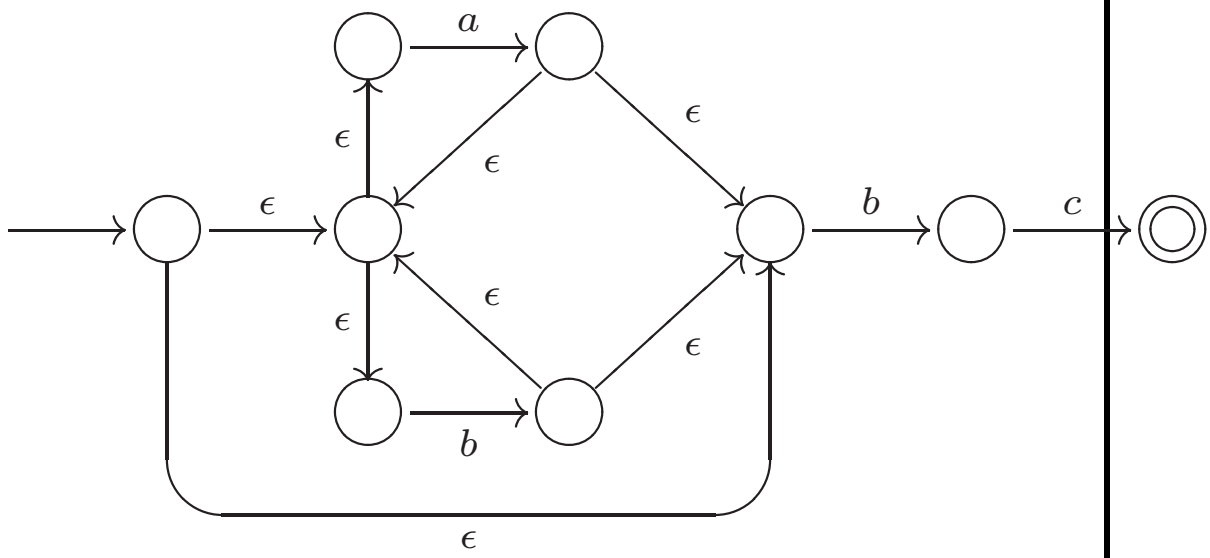
So $a \cup b$ yields:



Then $(a \cup b)^*$ yields:



Finally $(a \cup b)^*bc$ yields:



Of course there are simpler, equivalent automata.

DFAs $\Rightarrow$ REs

Lemma For every DFA D, there exists a regular expression E s.t. $L(E) = L(D)$.

The proof is based on the so-called **state elimination** construction (Lemma 1.60, pp.69–76).

There are two main steps:

1. Transform the DFA D into an equivalent GNFA G.
2. Repeatedly eliminate states from G until only two states remain.

A *generalised NFA (GNFA)* is an NFA in which transitions are labelled by REs. A GNFA follows a transition labelled by the RE E if the input has a prefix in $L(E)$.

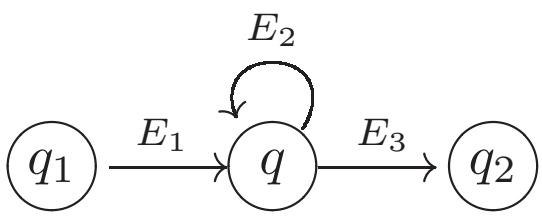
DFAs $\Rightarrow$ GNFAs

Given a DFA D, transform it to a GNFA G as follows:

1. If the initial state of D has any incoming transitions, add a new initial state q_0 with an ϵ-transition to the initial state of D.
2. Add a new accepting state q_A with an ϵ-transition from every accepting state of D to q_A.

State elimination

Consider a state q to be eliminated:



Replace every pair of transitions from q_1 through q to q_2 by a single transition with label $E_1 E_2^* E_3$, and eliminate q. (Note that $q_1 = q_2$ is possible. Note also that $\epsilon E = E\epsilon = E$.)

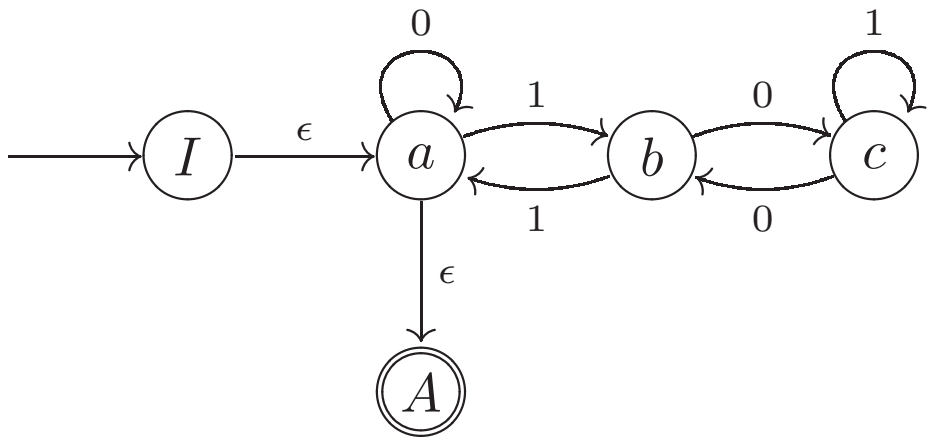
If this results in two transitions from q_1 to q_2 with labels E_1' and E_2', they should be merged into a single transition with label $E_1' + E_2'$.

Repeatedly eliminate intermediate states until no intermediate states remain.

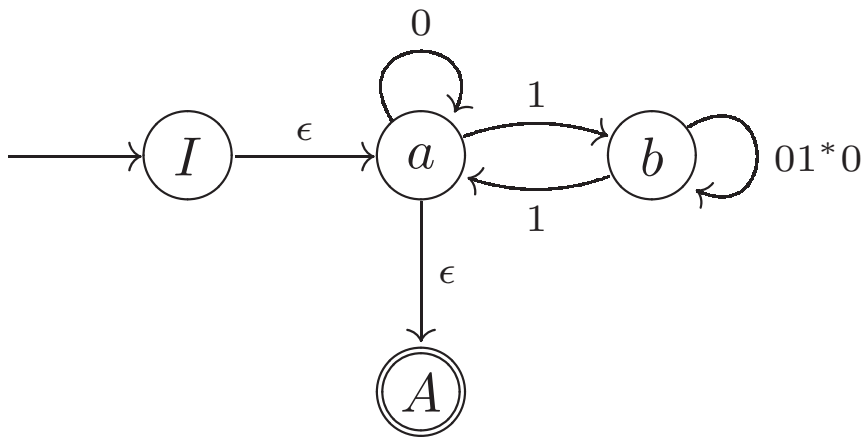
Return the RE E labelling the single transition from the initial state to the accepting state.

Example

Binary strings representing integers evenly divisible by 3 (after transforming to GNFA):

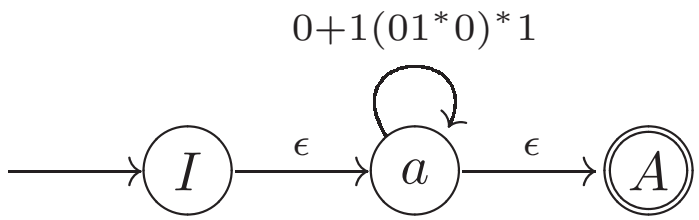


Eliminating state c:

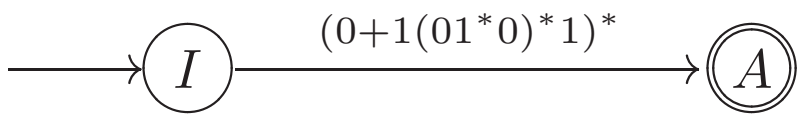


Example (cont.)

Eliminating state b:



Eliminating state a:



Final regular expression:

$$(0 + 1(01^*0)^*1)^*$$