

# Knowledge Forgetting in Circumscription: A Preliminary Report

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## Abstract

The theory of (variable) forgetting has received significant attention in nonmonotonic reasoning, especially, in answer set programming. However, the problem of establishing a theory of forgetting for some expressive nonmonotonic logics such as McCarthy’s circumscription is rarely explored. In this paper a theory of forgetting for propositional circumscription is proposed, which is not a straightforward adaption of existing approaches. In particular, some properties that are essential for existing proposals do not hold any longer or have to be reformulated. Several useful properties of the new forgetting are proved, which demonstrate suitability of the forgetting for circumscription. A sound and complete algorithm for the forgetting is developed and an analysis of computational complexity is given.

## Introduction

The ability of discarding irrelevant information from knowledge bases is a significant feature for logic-based agent systems (Lin and Reiter 1994; Lang, Liberatore, and Marquis 2003; Zhang and Zhou 2009). It can be naturally formalised as a form of *knowledge forgetting* in artificial intelligence (AI). Forgetting closely correlates with various notions in logics and AI, including irrelevance (Lake-meyer 1997), the weakest sufficient and strongest necessary conditions (Lin 2001; Doherty, Lukaszewicz, and Szalas 2001), uniform interpolation (Goranko and Otto 2007). Knowledge forgetting has been extensively applied in artificial intelligence (Lang, Liberatore, and Marquis 2003), such as cognitive robotics (Liu and Wen 2011; Rajaratnam et al. 2014), resolving conflict (Zhang and Foo 2006; Eiter and Wang 2008) and handling inconsistency (Lang and Marquis 2010), belief update (Delgrande, Jin, and Pelletier 2008), and ontology engineering (Lutz and Wolter 2011; Wang et al. 2014a).

Recent approaches to knowledge forgetting fall into two major streams. One stream covers works about forgetting for monotonic logical formalisms including propositional logic, predicate logic and its subclasses (typically, description logics), and (epistemic) modal logics, such as (Lang, Liberatore, and Marquis 2003; Lin and Reiter 1994;

Kontchakov, Wolter, and Zakharyashev 2008; Wang et al. 2010). Another stream is about forgetting for nonmonotonic logics, especially, logic programs under answer set semantics (Zhang and Foo 2006; Eiter and Wang 2008; Wang, Wang, and Zhang 2013; Wang et al. 2014b). Due to the pivotal role of commonsense reasoning in artificial intelligence, it is worthwhile to study forgetting in dominating nonmonotonic formalisms, such as circumscription (McCarthy 1980), default logic (Reiter 1980) and autoepistemic logic (Moore 1985).

For instance, a commonsense knowledge about *birds* is that they can fly unless there exists abnormality. Let us consider the following simple propositional case:

$$\Sigma = \left\{ \begin{array}{l} bird(tweety) \wedge \neg ab(tweety) \rightarrow fly(tweety), \\ bird(tweety) \end{array} \right\}.$$

In terms of circumscription<sup>1</sup> (Lifschitz 1994), one can use  $CIRC[\Sigma; ab; fly]$  to represent such a commonsense knowledge. It has a unique model  $\{bird(tweety), fly(tweety)\}$ . If  $bird(tweety)$  is discarded from  $\Sigma$  in the circumscription, a natural question is: what would be the resulting theory of circumscription? In propositional logic, the result of forgetting  $bird(tweety)$  is  $\psi \equiv \neg ab(tweety) \rightarrow fly(tweety) \equiv ab(tweety) \vee fly(tweety)$  for which the atom  $ab(tweety)$  is not circumscribed. Ideally, the result of forgetting  $bird(tweety)$  in  $\Sigma$  should be  $CIRC[\psi; ab, fly]$ , see Example 1.

As suggested by Delgrande (2015), the central ideal about forgetting is to retain exact “knowledge content” that is irrelevant from being forgotten signatures. From the perspective of knowledge content, the circumscription  $CIRC[\Sigma; ab; fly]$  specifies a knowledge base  $\Sigma$  as well. Thus, if  $\Sigma'$  is a result of discarding  $bird(tweety)$  from  $\Sigma$  in the circumscription then  $\Sigma'$  and  $\Sigma$  should be semantically as close as possible under circumscription.

Forgetting is known as an operator of quantifier elimination and thus in this way, it is closely related to circumscription. Some variants and (slight) generalizations of classical forgetting have been used to define and compute circumscription, e.g. literal forgetting (Lang, Liberatore, and Marquis 2003), literal forgetting with varying atoms (Moinard

<sup>1</sup>A propositional circumscription is a quantified Boolean formula, which is equivalent to a propositional formula. See the next section for a formal definition of circumscription.

2007), and projection and scope-determined circumscription (Wernhard 2012). However, none of these approaches considers a definition of forgetting for circumscription itself and it is not obvious to come up with such a suitable definition.

As a preliminary step towards knowledge forgetting in circumscription, we first introduce the concepts of strong equivalence and irrelevance for circumscription, and present some useful properties of them. Then we propose a notion of forgetting for circumscription, which takes into account the knowledge content of both circumscription and its knowledge base. This is not a straightforward extension of existing definitions, which can be seen from both the definition of forgetting and its major properties. The result of forgetting for circumscription result always exists and is unique up to the equivalence of propositional logic. To provide some further insights into the proposed theory of forgetting, we present several important properties of the forgetting, demonstrate that some properties holding for previous approaches do not hold any longer and explain why they should not hold.

Moreover, we present an algorithm for computing the result of forgetting for circumscription and present some preliminary results on computational complexity of the forgetting.

## Propositional Circumscription

We consider a propositional language  $\mathcal{L}^{\mathcal{A}}$  with a finite signature  $\mathcal{A}$  of propositional atoms (or, variables). Formulas of  $\mathcal{L}^{\mathcal{A}}$  are built from  $\mathcal{A}$ ,  $\top$  and  $\perp$  using the connectives  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$  as usual. For  $S \subseteq \mathcal{A}$ , we denote  $\bar{S} = \mathcal{A} \setminus S$  and  $\neg S = \{\neg p \mid p \in S\}$ .

An *interpretation* of  $\mathcal{L}^{\mathcal{A}}$  is identified as a set of atoms, which assigns *true* to each element in the set, and *false* to the others. A *model* of a formula is defined as usual. The set of all models of a formula  $\varphi$  is denoted  $\text{Mod}(\varphi)$ . Two formulas  $\varphi$  and  $\psi$  are *equivalent*, written  $\varphi \equiv \psi$ , if they have the same models. Given formulas  $\xi, \varphi, \psi$ , we write  $\xi[\varphi/\psi]$  for the formula obtained from  $\xi$  by replacing occurrences of  $\psi$  with  $\varphi$ . By  $\text{Var}(\varphi)$  we denote the set of atoms occurring in  $\varphi$ .

The expression  $\varphi(P, Q, Z)$  means that  $\varphi$  is a formula in  $\mathcal{L}^{\mathcal{A}}$  and  $\{P, Q, Z\}$  is a partition of  $\mathcal{A}$ . For simplicity,  $\varphi(P, Q, Z)$  can be just written as  $\varphi(P, Q)$  due to the fact that  $Z = \overline{P \cup Q}$ . For convenience we identify a tuple  $(t_1, \dots, t_n)$  of pairwise distinct objects with the set  $\{t_1, \dots, t_n\}$  when there is no confusion. This way, when  $X$  and  $Y$  are tuples of variables of the same length as  $P$  and  $Q$  respectively we use  $\varphi(X, Y, Z)$  to denote the formula obtained by simultaneously replacing all elements of  $P$  and  $Q$  in  $\varphi$  by the elements of  $X$  and  $Y$ , respectively.

For  $P = (p_1, \dots, p_k)$  and  $Q = (q_1, \dots, q_k)$ , we denote

$$\begin{aligned} P = Q & \text{ for } (p_1 \leftrightarrow q_1) \wedge \dots \wedge (p_k \leftrightarrow q_k); \\ P \leq Q & \text{ for } (p_1 \rightarrow q_1) \wedge \dots \wedge (p_k \rightarrow q_k); \\ P < Q & \text{ for } (P \leq Q) \wedge \neg(P = Q). \end{aligned}$$

In particular, if  $k = 0$  then both  $P = Q$  and  $P \leq Q$  are equivalent to  $\top$ ; while  $P < Q$  is equivalent to  $\perp$ .

**Definition 1** The circumscription of  $P$  in  $\varphi(P, Q, F)$  with atoms in  $Q$  being allowed to vary and the atoms in  $F$  being fixed, written  $\text{CIRC}[\varphi; P; Q]$ , is the following quantified formula

$$\varphi \wedge \neg \exists XY (\varphi(X, Y, F) \wedge X < P), \quad (1)$$

where  $X$  and  $Y$  are tuples of distinct variables having the same length as  $P$  and  $Q$ , respectively. For convenience,  $\text{CIRC}[\varphi; P; Q]$  is shortened as  $\text{CIRC}[\varphi; P]$  if  $Q = \emptyset$ .

A model of a circumscription is defined as that of the quantified Boolean formula. As it is well-known  $\exists p. \varphi \equiv \varphi(p/\perp) \vee \varphi(p/\top)$  and  $\forall p. \varphi \equiv \varphi(p/\perp) \wedge \varphi(p/\top)$ ,  $\text{CIRC}[\varphi; P; Q]$  is equivalent to a propositional formula.

Let  $M_i$  ( $i = 1, 2$ ) be interpretations and  $P, Q$  two disjoint sets of atoms. We say that  $M_1$  is *as least as*  $(P; Q)$ -preferred to  $M_2$ , written  $M_1 \leq^{P;Q} M_2$ , whenever it holds that

$$M_1 \cap \overline{P \cup Q} = M_2 \cap \overline{P \cup Q} \text{ and } M_1 \cap P \subseteq M_2 \cap P. \quad (2)$$

By  $M_1 <^{P;Q} M_2$  we mean  $M_1 \leq^{P;Q} M_2$  but not  $M_2 \leq^{P;Q} M_1$ . An interpretation  $M$  is  $[P; Q]$ -minimal with respect to (wrt) a collection  $\mathcal{M}$  of interpretations whenever there is no  $M' \in \mathcal{M}$  such that  $M' <^{P;Q} M$ . The following proposition is well-known (Lifschitz 1985).

**Proposition 1** An interpretation  $M$  is a model of circumscription  $\text{CIRC}[\varphi(P, Q); P; Q]$  iff  $M \models \varphi$  and  $M$  is  $[P; Q]$ -minimal wrt  $\text{Mod}(\varphi)$ .

The condition of minimality can be defined in a more general way. Let  $\mathcal{M}$  be a collection of interpretations,  $P$  and  $Q$  be two disjoint sets of atoms. We say that  $M$  is  $[P; Q]$ -minimal if every element of  $\mathcal{M}$  is  $[P; Q]$ -minimal wrt  $\mathcal{M}$ . By  $\text{MM}_{[P;Q]}(\mathcal{M})$  we denote the set of all  $[P; Q]$ -minimal elements of  $\mathcal{M}$ ;  $\text{MM}(\mathcal{M})$  denotes the set of minimal (under set inclusion) ones in  $\mathcal{M}$ . For a formula  $\varphi$ , by  $\text{MM}_{[P;Q]}(\varphi)$  (resp.  $\text{MM}(\varphi)$ ) we denote  $\text{MM}_{[P;Q]}(\text{Mod}(\varphi))$  (resp.  $\text{MM}(\text{Mod}(\varphi))$ ).

Given a collection  $\mathcal{M}$  of interpretations, by  $\text{Form}(\mathcal{M}, \mathcal{A})$  we mean the following formula (over the signature  $\mathcal{A}$ ):

$$\bigvee_{M \in \mathcal{M}} \left( \bigwedge (M \cup \neg(\overline{M})) \right) \quad (3)$$

In particular,  $\text{Form}(\mathcal{M}, \mathcal{A}) \equiv \perp$  if  $\mathcal{M} = \emptyset$ , and  $\text{Form}(\mathcal{M}, \mathcal{A}) \equiv \top$  if  $\mathcal{A} = \emptyset^2$  and  $\mathcal{M} = \{\emptyset\}$ . When the underlying signature  $\mathcal{A}$  is clear from its context, we shorten  $\text{Form}(\mathcal{M}, \mathcal{A})$  as  $\text{Form}(\mathcal{M})$ .

**Lemma 1** Let  $\mathcal{M}$  be a collection of interpretations and  $P, Q$  two disjoint sets of atoms. Then there is a formula  $\varphi$  s.t  $\text{Mod}(\text{CIRC}[\varphi; P; Q]) = \mathcal{M}$  iff  $\mathcal{M}$  is  $[P; Q]$ -minimal.

By the definition of circumscription, it is easy to see the following two properties.

**Proposition 2** Let  $\varphi(P, Q)$  and  $\psi(P, Q)$  be two formulas.

- (i)  $\text{CIRC}[\varphi; \emptyset; Q] \equiv \text{CIRC}[\varphi; \emptyset; \emptyset] \equiv \varphi$ .
- (ii) If  $\varphi \equiv \psi$  then  $\text{CIRC}[\varphi; P; Q] \equiv \text{CIRC}[\psi; P; Q]$ .

<sup>2</sup>Please note that  $2^{\mathcal{A}} = \{\emptyset\}$  if  $\mathcal{A} = \emptyset$ .

## Strong Equivalence and Irrelevance

In this section we adapt two important concepts *strong equivalence* and *relevane* to circumscription. The former is needed in several application domains such as knowledge reuse since it is necessary to guarantee that when a module in a knowledge base (KB) is replaced with an ‘equivalent’ module, the resulting KB is still ‘equivalent’ to the original KB. The latter is closely relevant to knowledge forgetting.

### Strong equivalence

Two formulas  $\varphi(P, Q)$  and  $\psi(P, Q)$  are called

- (1)  $[P; Q]$ -*equivalent* if  $\text{CIRC}[\varphi; P; Q] \equiv \text{CIRC}[\psi; P; Q]$ , denoted by  $\varphi \equiv_{\text{circ}[P; Q]} \psi$ ;
- (2)  $\varphi$  and  $\psi$  are *strongly*  $[P; Q]$ -*equivalent*, written  $\varphi \equiv_{\text{circ}[P; Q]}^s \psi$ , if  $\xi \wedge \varphi \equiv_{\text{circ}[P; Q]} \xi \wedge \psi$  for any formula  $\xi$ .

The following result shows that the above definition of strong equivalence is sufficient for the purpose of knowledge module reuse.

**Proposition 3** *Let  $\varphi(P, Q)$  and  $\psi(P, Q)$  be two formulas where  $P$  and  $Q$  are disjoint tuples of atoms. Then  $\varphi$  and  $\psi$  are strongly  $[P; Q]$ -equivalent iff  $\xi \equiv_{\text{circ}[P; Q]} \xi[\varphi/\psi]$  for any formula  $\xi$ .*

**Proof sketch:** We write  $\varphi \equiv_{\text{circ}[P; Q]}^s \psi$  if  $\xi \equiv_{\text{circ}[P; Q]} \xi[\varphi/\psi]$  holds for any formula  $\xi$ ; It is easy to show that  $\varphi \equiv \psi$  implies  $\varphi \equiv_{\text{circ}[P; Q]}^s \psi$ , and  $\varphi \equiv_{\text{circ}[P; Q]}^s \psi$  implies  $\varphi \overset{s}{\equiv}_{\text{circ}[P; Q]} \psi$ . If  $\varphi \overset{s}{\equiv}_{\text{circ}[P; Q]} \psi$  but  $\varphi \not\equiv \psi$  then we can construct a formula  $\xi = \bigwedge (M \cup \neg \bar{M})$  such that  $M \models \varphi$  and  $M \not\models \psi$ , or  $M \models \psi$  and  $M \not\models \varphi$ . Either case implies  $\xi \wedge \varphi \not\equiv_{\text{circ}[P; Q]} \xi \wedge \psi$ , viz  $\varphi \overset{s}{\equiv}_{\text{circ}[P; Q]} \psi$  does not hold, a contradiction. ■

By the definition of strong equivalence, we have the following simple but useful result.

**Proposition 4** *If two propositional formulas  $\varphi$  and  $\psi$  are equivalent in propositional logic, then they are strongly equivalent in circumscription.*

### Irrelevance

Let  $V \subseteq \mathcal{A}$ . Recall that a formula  $\varphi$  is *irrelevant* to  $V$  in propositional logic, written  $IR(\varphi, V)$ , if there exists a formula  $\psi$  containing no atoms from  $V$  such that  $\varphi \equiv \psi$ . This notion of irrelevance can be easily generalised to the circumscription as shown in the next definition.

For convenience, a singleton  $\{E\}$  is identified with  $E$  when it is clear from its context.

**Definition 2** ( $[P; Q]$ -*irrelevance*) *Let  $V \subseteq \mathcal{A}$ . A formula  $\varphi(P, Q)$  is  $[P; Q]$ -irrelevant from  $V$ , written  $IR_{[P; Q]}(\varphi, V)$ , if there is a formula  $\psi$  containing no atoms from  $V$  such that  $\text{CIRC}[\varphi; P; Q] \equiv \text{CIRC}[\psi; P; Q]$ .*

It is easy to see that  $IR(\varphi, V)$  implies  $IR_{[P; Q]}(\varphi, V)$ , but not vice versa. For instance, over the signature  $\{p, q\}$ ,  $\text{CIRC}[\neg p; \{p, q\}] \equiv \text{CIRC}[\neg q; \{p, q\}] \equiv \text{CIRC}[\top; \{p, q\}]$ , thus both  $\neg p$  and  $\neg q$  are  $[\{p, q\}; \emptyset]$ -irrelevant from  $p$  and  $q$ . However,  $IR(\neg p, p)$  does not hold.

**Proposition 5**  *$IR_{[P; Q]}(\varphi, V)$  implies  $IR(\varphi, V)$  if and only if  $MM_{[P; Q]}(\varphi) = \text{Mod}(\varphi)$ .*

It shows that, whenever all models of  $\varphi$  are  $[P; Q]$ -minimal wrt  $\text{Mod}(\varphi)$ ,  $IR_{[P; Q]}(\varphi, V)$  is identical with  $IR(\varphi, V)$ .

It would be difficult to determine if a formula  $\varphi(P, Q)$  is  $[P; Q]$ -irrelevant from  $V$  using Definition 2. To provide an alternative characterisation of  $[P; Q]$ -irrelevance, we need the following two lemmas.

Two sets  $M$  and  $N$  of atoms are  $V$ -*bisimilar*, denoted by  $M \sim_V N$ , if  $M \setminus V = N \setminus V$ .

**Lemma 2** *Let  $\varphi(P, Q)$  be a formula and  $V \subseteq \mathcal{A}$  with  $V \cap (P \cup \text{Var}(\varphi)) = \emptyset$ . Then  $M \models \text{CIRC}[\varphi; P; Q]$  iff  $M' \models \text{CIRC}[\varphi; P; Q]$  for each  $M' \sim_V M$ .*

This lemma shows that when checking the validity of  $IR_{[P; Q]}(\varphi, V)$ , we need only consider one representative among all  $V$ -bisimilar interpretations. We then provide some characterisations of  $[P; Q]$ -irrelevance from a single atom.

**Lemma 3** *Let  $\varphi(P, Q, F)$  be a formula and  $p \in \mathcal{A}$ .*

- (i) *If  $p \in Q \cup F$  then  $\varphi$  is  $[P; Q]$ -irrelevant from  $p$  iff  $M \models \text{CIRC}[\varphi; P; Q]$  implies  $M' \models \text{CIRC}[\varphi; P; Q]$  for each pair of interpretations  $M$  and  $M'$  with  $M' \sim_p M$ .*
- (ii) *For  $p \in P$ ,  $\varphi$  is  $[P; Q]$ -irrelevant from  $p$  iff  $\text{CIRC}[\varphi; P; Q] \models \neg p$ .*
- (iii) *If  $V' \subseteq V \subseteq \mathcal{A}$ , then  $IR_{[P; Q]}(\varphi, V)$  implies  $IR_{[P; Q]}(\varphi, V')$ .*

Now we are ready to provide a model-theoretic characterisation of  $[P; Q]$ -irrelevance.

**Theorem 1** (**Characterization of  $[P; Q]$ -irrelevance**) *A formula  $\varphi$  is  $[P; Q]$ -irrelevant from a set  $V$  of atoms iff the following two conditions hold:*

- (i)  $\text{CIRC}[\varphi; P; Q] \models \bigwedge \neg(V \cap P)$ , and
- (ii)  $M \models \text{CIRC}[\varphi; P; Q]$  implies  $M' \models \text{CIRC}[\varphi; P; Q]$  for each pair of interpretations  $M$  and  $M'$  with  $M' \sim_{V \setminus P} M$ .

**Proof sketch:**  $(\Rightarrow)$  The condition (i) follows from (ii) of Lemma 3, while the condition (ii) follows from Lemma 2.

$(\Leftarrow)$  Let  $\psi = \text{Form}(\text{Mod}(\text{CIRC}[\varphi; P; Q]), \mathcal{A})$ . By (iii) of Proposition 2, it holds  $\text{CIRC}[\text{CIRC}[\psi; P; Q]; P; Q] \equiv \text{CIRC}[\psi; P; Q] \equiv \psi$ . We can show that, under the conditions (i) and (ii),  $IR(\psi, V)$ . ■

The  $[P; Q]$ -irrelevance of a formula from a union of two sets of atoms is reduced to the irrelevance from each set.

**Proposition 6** *Let  $V_1$  and  $V_2$  be sets of atoms, and  $\varphi(P, Q)$  be a formula. Then  $IR_{[P; Q]}(\varphi; V_1 \cup V_2)$  iff  $IR_{[P; Q]}(\varphi, V_i)$  for  $i = 1, 2$ .*

**Proof sketch:**  $(\Rightarrow)$  By (i) of Theorem 1,  $\text{CIRC}[\varphi; P; Q] \models \bigwedge \neg((V_1 \cup V_2) \cap P)$  due to  $\text{CIRC}[\varphi; P; Q] \models \bigwedge \neg(V_i \cap P)$  ( $i = 1, 2$ ). By (ii) of Theorem 1,  $M \models \text{CIRC}[\varphi; P; Q]$  implies, for every  $M' \sim_{V_1 \setminus P} M$ ,  $M' \models \text{CIRC}[\varphi; P; Q]$  due to  $IR_{[P; Q]}(\varphi, V_1)$ , which implies  $M'' \models \text{CIRC}[\varphi; P; Q]$  for every  $M'' \sim_{V_2 \setminus P} M'$  by  $IR_{[P; Q]}(\varphi, V_2)$ , i.e.  $M'' \models \text{CIRC}[\varphi; P; Q]$  for each  $M'' \sim_{(V_1 \cup V_2) \setminus P} M$ .

( $\Leftarrow$ ) It follows from (iii) of Lemma 3.  $\blacksquare$

By the above proposition, a formula is  $[P; Q]$ -irrelevant from a set  $V \subseteq \mathcal{A}$  if and only if it is  $[P; Q]$ -irrelevant from each element in  $V$ .

### Knowledge Forgetting in Circumscription

Let  $S$  and  $V$  be two subsets of  $\mathcal{A}$  and  $\mathcal{S} \subseteq 2^{\mathcal{A}}$ . We define:

- The *exclusion* of  $S$  on  $V$ , written  $S_{\dagger V}$ , is the set  $S \setminus V$ ;
- The *projection* of  $S$  on  $V$ , written  $S|_V$ , is the set  $S \cap V$ ;
- The *expansion* of  $S$  on  $V$ , written  $S_{\dagger V}$ , is the collection  $\{S' \mid S' \sim_V S\}$ ;
- $\mathcal{S}_{\dagger V} = \{S_{\dagger V} \mid S \in \mathcal{S}\}$ ;
- $\mathcal{S}|_V = \{S|_V \mid S \in \mathcal{S}\}$ ;
- $\mathcal{S}_{\dagger V} = \bigcup_{S \in \mathcal{S}} S_{\dagger V}$ .

Recall that in propositional logic, a formula  $\psi$  is a *result* of forgetting  $V \subseteq \mathcal{A}$  from a formula  $\varphi$  iff  $\text{Mod}(\psi) = \text{Mod}(\varphi)_{\dagger V}$  (Lang, Liberatore, and Marquis 2003). A syntactic counterpart is the quantified formula  $\exists V.\varphi$ . Its representation theorem says that  $\exists V.\varphi$  consists of the knowledge/information of  $\varphi$  which is irrelevant to  $V$ , i.e.  $\exists V.\varphi \equiv \{\xi \mid \text{IR}(\xi, V) \text{ and } \varphi \models \xi\}$ . We denote the result of classical forgetting  $V$  in  $\varphi$  by  $F(\varphi, V)$ .

**Definition 3** Let  $\varphi(P, Q)$  be a formula and  $V \subseteq \mathcal{A}$ . A formula  $\psi$  is a result of forgetting  $V$  in  $\text{CIRC}[\varphi; P; Q]$ , or  $[P; Q]$ -forgetting  $V$  in  $\varphi$ , if the following two conditions hold:

- $\text{IR}(\psi, V)$ , and
- $\text{Mod}(\psi)$  is a maximal subset set of  $\text{Mod}(\varphi)_{\dagger V}$  such that,

$$MM_{[P; Q]}(\psi)_{\dagger V} = MM_{[P; Q]}(\text{Mod}(\text{CIRC}[\varphi; P; Q]))_{\dagger V}. \quad (4)$$

Condition (a) requires that the result  $\psi$  of forgetting for circumscription is irrelevant to  $V$ , which is standard for all definitions of forgetting. However, condition (b) is not obvious and some elaboration on it is necessary. A major condition for a definition of forgetting is that the result of forgetting should preserve the consequences of original theory that are irrelevant to the signature to be forgotten. Given condition (a), without loss of generality, let us assume that  $\psi$  does not contain any symbols in  $V$ . Then for condition (b),  $\text{Mod}(\psi) \subseteq \text{Mod}(\varphi)_{\dagger V}$  is to guarantee that every consequence  $\xi$  of  $[P; Q]$ -forgetting  $V$  in  $\varphi$  is entailed by  $\psi$  if  $\xi$  is  $[P; Q]$ -irrelevant to  $V$ . while the maximality of  $\text{Mod}(\psi)$  assures a theory closest to  $\varphi$  is obtained. Eq.(4) is the major part of condition (b). We could have  $\text{Mod}(\text{CIRC}[\psi; P; Q]) = \text{Mod}(\text{CIRC}[\varphi; P; Q])_{\dagger V}$  instead of Eq.(4). However, this is infeasible. The reason is that some interpretations in  $\text{Mod}(\text{CIRC}[\varphi; P; Q])_{\dagger V}$  may be comparable when allowing atoms in  $Q$  varied; but the models of  $\text{CIRC}[\psi; P; Q]$  (equivalently, the  $[P; Q]$ -minimal models of  $\psi$ ) must be incomparable. This is why we take  $MM_{[P; Q]}(\text{Mod}(\text{CIRC}[\varphi; P; Q]))_{\dagger V}$  instead of simpler  $\text{Mod}(\text{CIRC}[\varphi; P; Q])_{\dagger V}$ .

Note that a result of  $[P; Q]$ -forgetting  $V$  is only needed to be semantically irrelevant to  $V$  and thus it may not be syntactically irrelevant. That is, a result of  $[P; Q]$ -forgetting  $V$

may still contain variables in  $V$ . Moreover, one might wish to require that a result of  $[P; Q]$ -forgetting  $V$  does not contain symbols in  $V$ . This is possible and we just need to replace (4) with the following equation:

$$MM_{[P', Q']}(\psi)_{\dagger V} = MM_{[P', Q']}(\text{Mod}(\text{CIRC}[\varphi; P; Q]))_{\dagger V} \quad (5)$$

where  $P' = P_{\dagger V}$  and  $Q' = Q_{\dagger V}$ .

However, we can show that these two definitions are equivalent in the sense that (1) each result  $\xi$  of forgetting defined by Eq.(4) is that defined by Eq.(5) if  $\xi$  does not contain variables to be forgotten; and (2) conversely, each result of forgetting defined by Eq.(5) is that defined by Eq.(4).

**Example 1** Consider circumscription theory  $\text{CIRC}[\Sigma; P; Q]$  in Introduction where  $P = \{ab(\text{tweety})\}$  and  $Q = \{fly(\text{tweety})\}$ . Then  $\psi \equiv ab(\text{tweety}) \vee fly(\text{tweety})$  is a result of  $[P; Q]$ -forgetting  $bird(\text{tweety})$  in  $\text{CIRC}[\Sigma; P; Q]$ , which can be seen from  $\psi \equiv \exists V.\Sigma \equiv \exists V.\text{CIRC}[\Sigma; P; Q]$ .

**Theorem 2** For any formula  $\varphi(P, Q)$  and any  $V \subseteq \mathcal{A}$ , there always exists a result of forgetting  $V$  in  $\text{CIRC}[\varphi; P; Q]$ .

**Proof sketch:** A result of forgetting  $V$  in  $\text{CIRC}[\varphi; P; Q]$  is constructed as follows.

$$\left. \begin{aligned} \mathcal{M}_1 &= \text{Mod}(\text{CIRC}[\varphi; P; Q])_{\dagger V}, \\ \mathcal{M}_2 &= \{M_{\dagger V} \mid M \in \text{Mod}(\varphi)_{\dagger V} \text{ and } \exists M' \in \mathcal{M}_1 \\ &\quad \text{such that } M' <^{P; Q} M_{\dagger V}\}, \\ \mathcal{M} &= \mathcal{M}_1 \cup \mathcal{M}_2, \\ \psi &= \text{Form}(\mathcal{M}, \bar{V}). \end{aligned} \right\} \quad (6)$$

It is obvious that  $\text{Mod}(\psi) \subseteq \text{Mod}(\varphi)_{\dagger V}$  and  $\text{IR}(\psi, V)$ .

Let  $\mathcal{M}' = \mathcal{M}_{\dagger V}$  and  $\psi' = \text{Form}(\mathcal{M}', \mathcal{A})$ . On the one hand, we can see that  $\psi \equiv \psi'$  and  $MM_{[P; Q]}(\mathcal{M}_1)_{\dagger V} = MM_{[P; Q]}(\mathcal{M})_{\dagger V} = MM_{[P; Q]}(\mathcal{M}')_{\dagger V}$ . On the other hand, the maximality of  $\mathcal{M}'$  follows from the construction and the fact that  $\text{IR}(\psi, V)$ .  $\blacksquare$

The above theorem shows that  $[P; Q]$ -forgetting results are unique up to equivalence in propositional logic (and thus strongly equivalent in circumscription by Proposition 4). For this reason, by  $F_{[P; Q]}(\varphi, V)$  we denote an arbitrary result of forgetting  $V$  in  $\text{CIRC}[\varphi; P; Q]$ .

From the proof of Theorem 2, we have the following corollary.

**Corollary 3** Let  $\psi$  be a result of  $[P; Q]$ -forgetting  $V \subseteq \mathcal{A}$  in  $\varphi(P, Q)$ . Then

- $\exists V.\text{CIRC}[\varphi; P; Q] \models \psi$  and  $\psi \models \exists V.\varphi$ ,
- $\text{CIRC}[\psi; P; Q] \equiv \text{CIRC}[\psi; P_{\dagger V}; Q_{\dagger V}] \wedge \bigwedge \neg(V \cap P)$ .

According to Theorem 2 in (Lee and Lin 2006),  $\text{CIRC}[\varphi; P; Q]$  is equivalent to  $\varphi \wedge \varphi'$  where  $\varphi'$  is the conjunction of the conjunctive loop formulas for  $\text{CIRC}[\varphi; P; Q]$ . Thus, item (i) of this corollary shows a relationship between definitions of forgetting for circumscription and for propositional logic due to  $\exists V.\text{CIRC}[\varphi; P; Q] \equiv \exists V.(\varphi \wedge \varphi') \equiv F(\varphi \wedge \varphi', V)$ .

**Example 2** Consider  $\varphi = p \vee q$  over the signature  $\{p, q\}$ , and  $V = \{p\}$ . Then  $\text{Mod}(\varphi) = \{\{p\}, \{q\}, \{p, q\}\}$ ,  $\text{Mod}(\varphi)_{\dagger V} = \text{Mod}(\varphi) \cup \{\emptyset\}$  and,  $F(\varphi, V) \equiv \top$ .

- (1) Let  $P = \{p\}$  and  $Q = \emptyset$ . Since  $\text{CIRC}[\varphi; P; Q] \equiv (p \vee q) \wedge \neg(p \wedge q)$  whose models are  $\{p\}$  and  $\{q\}$ , we have  $\text{Mod}(\text{CIRC}[\varphi; P; Q])_{\uparrow V} = \{\emptyset, \{q\}\}$ . By Eq. (6),  $\mathcal{M} = \{\emptyset, \{q\}\}$ , which is  $[P; Q]$ -minimal. Thus,

$$\psi = \text{Form}(\mathcal{M}, \bar{V}) = q \vee \neg q \equiv \top \equiv F_{[P; Q]}(\varphi; V).$$

It is evident that  $\varphi \models \psi$  but  $\psi \not\models \varphi$ .

- (2) Let  $P = \{p, q\}$  and  $Q = \emptyset$ . One can verify that  $\text{CIRC}[\varphi; P; Q] \equiv (p \vee q) \wedge \neg(p \wedge q)$  and  $\text{Mod}(\text{CIRC}[\varphi; P; Q])_{\uparrow V} = \{\emptyset, \{q\}\}$ , which is not  $[P; Q]$ -minimal due to  $\emptyset <^{P; Q} \{q\}$ . According to the construction of  $\mathcal{M}$  in Eq. (6), one can verify that  $F_{[P; Q]}(\varphi, V) \equiv \top$ .
- (3) Let  $P = \{p\}$  and  $Q = \{q\}$ . Then  $\text{CIRC}[\varphi; P; Q] \equiv \neg p \wedge q$  and  $\text{Mod}(\text{CIRC}[\varphi; P; Q])_{\uparrow V} = \{\{q\}\}$ . By Eq.(6)  $\mathcal{M} = \{\{q\}\}$ , which is  $[P; Q]$ -minimal, then  $\psi = \text{Form}(\mathcal{M}, \bar{V}) = q$ .  
It is obvious  $\psi \models \varphi$  but  $\varphi \not\models \psi$ . Recall that  $F(\varphi, p) \equiv \top$ . Thus,  $\psi \models F(\varphi, p)$  but the reverse does not hold.

In the next proposition, item (i) shows that  $[P; Q]$ -forgetting is a generalization of the forgetting for propositional logic; item (ii) states that results of  $[P; Q]$ -forgetting in two equivalent knowledge bases are still equivalent.

**Proposition 7** Let  $\varphi(P, Q)$  be a formula and  $V \subseteq \mathcal{A}$ .

- (i) If  $P = \emptyset$  then  $F_{[P; Q]}(\varphi, V) \equiv F(\varphi, V)$ .
- (ii) If  $\varphi \equiv \psi$  then  $F_{[P; Q]}(\varphi, V) \equiv F_{[P; Q]}(\psi, V)$ .
- (iii)  $F_{[P; Q]}(\varphi, V) \equiv \exists V. \text{CIRC}[\varphi; P; Q]$  whenever  $\text{Mod}(\varphi)$  is  $[P; Q]$ -minimal.

**Proposition 8** Let  $\varphi(P, Q)$  be a formula and  $V \subseteq \mathcal{A}$ . It holds that  $IR(\varphi, V)$  iff  $\varphi \equiv F_{[P; Q]}(\varphi, V)$ .

**Proof sketch:** ( $\Rightarrow$ ) By  $IR(\varphi, V)$ , if  $M_{\uparrow V} \models \varphi$ , then  $M_{\uparrow V}$  is  $[P; Q]$ -minimal wrt  $\text{Mod}(\text{CIRC}[\varphi; P; Q])$  iff it is  $[P; Q]$ -minimal wrt  $\text{Mod}(\text{CIRC}[\varphi; P; Q])_{\uparrow V}$ . Therefore,  $\mathcal{M} = \text{Mod}(\varphi)_{\uparrow V}$ , where  $\mathcal{M}$  is defined as in Eq.(6). ■

As  $IR(\varphi, V)$  implies  $IR_{[P; Q]}(\varphi, V)$ , this proposition shows that the results of  $[P; Q]$ -forgetting  $V$  from  $\varphi$  are  $[P; Q]$ -irrelevant from  $V$ , i.e.  $IR_{[P; Q]}(F_{[P; Q]}(\varphi, V), V)$ .

### Properties that are different from classical forgetting

In this subsection, through some examples we will demonstrate that our new theory of forgetting is distinct from classical forgetting and previous approaches to forgetting in answer set programming.

First, if the original formula is Horn, the  $[P; Q]$ -forgetting result may not be Horn expressible. This can be seen from the following example. This is a difference of  $[P; Q]$ -forgetting from classical forgetting and (knowledge) forgetting in answer set programming (Wang et al. 2014b; Wang, Wang, and Zhang 2013).

**Example 3** Let  $\varphi = p \wedge s \wedge \neg q \vee p \wedge s \wedge \neg r \vee \neg(p \vee q \vee s \vee r)$  over the signature  $\{p, q, r, s\}$ . Then

$$\text{Mod}(\varphi) = \{\emptyset, \{p, s\}, \{p, q, s\}, \{p, r, s\}\}.$$

It is obvious that  $\varphi$  is Horn expressible, i.e., there is a Horn formula  $\varphi'$  such that  $\varphi \equiv \varphi'$ , as  $\text{Mod}(\varphi)$  is closed under intersection. For  $P = \{p\}$  and  $Q = \{s\}$ , we can see that  $\text{Mod}(\text{CIRC}[\varphi; P; Q]) = \{\emptyset, \{p, q, s\}, \{p, r, s\}\}$ . Here  $\{p, s\} \not\models \text{CIRC}[\varphi; P; Q]$  is due to  $\emptyset <^{P; Q} \{p, s\}$ . For  $V = \{p\}$ , by Eq. (6), we have that, over the signature  $\{q, r, s\}$ ,

$$\mathcal{M} = \text{Mod}(F_{[P; Q]}(\varphi, V)) = \{\emptyset, \{q, s\}, \{r, s\}\}.$$

The reason for  $\{p, s\}_{\uparrow V} = \{s\} \notin \mathcal{M}$  is that  $\{s\}$  is  $[P; Q]$ -minimal wrt  $\text{Mod}(\text{CIRC}[\varphi; P; Q])_{\uparrow V}$ , and it is not in  $\text{Mod}(\text{CIRC}[\varphi; P; Q])_{\uparrow V}$ . Thus,  $F_{[P; Q]}(\varphi, V)$  is not Horn expressible.

The next example indicates that in general,  $F_{[P; Q]}(\varphi, V_1 \cup V_2)$  may not entail  $F_{[P; Q]}(F_{[P; Q]}(\varphi, V_1), V_2)$ . Therefore,  $[P; Q]$ -forgetting is sensitive to the orders of being forgotten atoms.

**Example 4** Let  $\varphi = p \wedge \neg q \wedge \neg r \vee \neg p \wedge q \wedge r$ ,  $V_1 = \{p\}$ ,  $V_2 = \{q\}$ ,  $P = \{q\}$  and  $Q = \{r\}$ . We have the following:

- $\psi = F_{[P; Q]}(\varphi, V_1 \cup V_2) \equiv \top$ .
- $\psi_1 = F_{[P; Q]}(\varphi, V_1) \equiv q \wedge r \vee \neg q \wedge \neg r$ .
- $\psi_2 = F_{[P; Q]}(\varphi, V_2) \equiv p \wedge \neg r \vee \neg p \wedge r$ .
- $F_{[P; Q]}(\psi_1, V_2) \equiv \neg r$  and  $F_{[P; Q]}(\psi_2, V_1) \equiv \top$ .

The following example shows that  $V_1 \subseteq V_2$  does not imply  $F_{[P; Q]}(\varphi, V_1) \models F_{[P; Q]}(\varphi, V_2)$ . Thus, the operator of  $[P; Q]$ -forgetting  $V$  is nonmonotonic wrt  $V$ .

**Example 5** Let  $\varphi = p \wedge q \wedge r \vee p \wedge \neg q \wedge \neg r$ ,  $P = \{p, q\}$ ,  $Q = \{r\}$ ,  $V_1 = \{p\}$  and  $V_2 = \{p, q\}$ . Then

- $\text{Mod}(\varphi) = \{\{p\}, \{p, q, r\}\}$ ,  $\text{MM}_{[P; Q]}(\varphi) = \{\{p\}\}$ .
- $F_{[P; Q]}(\varphi, V_1) \equiv q \wedge r \vee \neg q \wedge \neg r$ .
- $F_{[P; Q]}(\varphi, V_2) \equiv \neg q \wedge \neg r$ .

It is evident that  $\{q, r\}$  is a model of  $F_{[P; Q]}(\varphi, V_1)$ . However, it is not a model of  $F_{[P; Q]}(\varphi, V_2)$ .

For the forgetting in propositional logic, it holds that  $F(\varphi \vee \psi, V) \equiv F(\varphi, V) \vee F(\psi, V)$ . This property does not hold for  $[P; Q]$ -forgetting as shown by the next example.

**Example 6** Let  $\xi = \varphi \vee \psi$  where  $\varphi = p$  and  $\psi = q$ ,  $P = \{p\}$ ,  $Q = \{q\}$  and  $V = \{p\}$ . We have the following:

- $\text{CIRC}[\xi; P; Q] \equiv \neg p \wedge q$ ,  $\text{CIRC}[\varphi; P; Q] \equiv p$  and  $\text{CIRC}[\psi; P; Q] \equiv \neg p \wedge q$ .
- $F_{[P; Q]}(\varphi, V) \equiv \top$ ,  $F_{[P; Q]}(\psi, V) \equiv q$ .
- $F_{[P; Q]}(\varphi, V) \vee F_{[P; Q]}(\psi, V) \equiv \top$ .
- $F_{[P; Q]}(\varphi \vee \psi, V) \equiv q$ .

The following example shows that  $F_{[P; Q]}(\varphi \wedge \psi, V)$  is not equivalent to  $F_{[P; Q]}(\varphi, V) \wedge \psi$  even though  $IR(\psi, V)$  holds.

**Example 7** Let  $\xi = \varphi \wedge \psi$  where  $\varphi = p$  and  $\psi = q$ ,  $P = \{p, q\}$ ,  $Q = \emptyset$ . We have the following:

- $\text{CIRC}[\xi; P; Q] \equiv p \wedge q$ ,  $\text{CIRC}[\varphi; P; Q] \equiv p \wedge \neg q$ .
- $F_{[P; Q]}(\xi, p) \equiv q$ .
- $F_{[P; Q]}(\varphi, p) \equiv \neg q$ ,  $F_{[P; Q]}(\varphi, p) \wedge \psi \equiv \perp$ .

While several properties of classical forgetting do not hold for  $[P; Q]$ -forgetting, in the next subsection we identify conditions under which these properties hold.

## Further properties of $[P; Q]$ -forgetting

In this section we present some further properties of  $[P; Q]$ -forgetting.

Let  $\mathcal{M} \subseteq 2^{\mathcal{A}}$  and  $F \subseteq \mathcal{A}$ . We define a binary relation  $\simeq_F$  on  $\mathcal{M}$  as  $M_1 \simeq_F M_2$  iff  $M_1 \cap F = M_2 \cap F$  where  $M_1, M_2 \in \mathcal{M}$ . It is evident that  $\simeq_F$  is reflexive, symmetric and transitive, and thus it is an equivalence relation. For  $x \in \mathcal{M}$ , by  $[x] = \{N \in \mathcal{M} | N \simeq_F x\}$  we denote the *equivalence class* to which  $x$  belongs. The set of all equivalence classes of  $\mathcal{M}$  by  $\simeq_F$ , denoted  $\mathcal{M}/\simeq_F$ , is the *quotient set* of  $\mathcal{M}$  by  $\simeq_F$ , i.e.  $\mathcal{M}/\simeq_F = \{[x] | x \in \mathcal{M}\}$ . So  $\mathcal{M}/\simeq_F$  is a partition of  $\mathcal{M}$ . It is easy to see that  $\text{CIRC}[\varphi; P; Q]$  has a model in  $[x]$  for each  $[x] \in \text{Mod}(\varphi)/\simeq_F$  where  $F = P \cup Q$ .

Before we present some properties of forgetting for circumscription, we observe two lemmas.

**Lemma 4** *Let  $\varphi(P, Q)$  be a formula and  $M$  a model of  $\varphi$ .*

- (i)  $M \in \text{MM}_{[P; Q]}(\varphi)$  iff  $M|_P \in \text{MM}([M]|_P)$ .
- (ii) If  $V \subseteq Q$  or  $P \subseteq V \subseteq \mathcal{A}$ , then  $M \in \text{MM}_{[P; Q]}(\varphi)$  iff  $M|_V \in \text{MM}_{[P; Q]}(\varphi)|_V$ .

Please note that in item (ii) of the lemma, condition  $V \subseteq Q$  cannot be weakened to  $V \subseteq F \cup Q$ .

**Lemma 5** *Let  $\varphi(P, Q)$  be a formula,  $V \subseteq \mathcal{A}$  and  $\psi \equiv \text{F}_{[P; Q]}(\varphi, V)$ .*

- (i)  $\text{MM}_{[P; Q]}(\psi)|_F = (\text{MM}_{[P; Q]}(\varphi)|_V)|_F$ .
- (ii) If  $V \subseteq Q$  or  $P \subseteq V$  then  $\text{MM}_{[P; Q]}(\varphi)|_V = \text{MM}_{[P; Q]}(\psi)|_V$ .
- (iii) If  $V \subseteq F \cup Q$  then  $\text{Mod}(\psi) = \text{Mod}(\varphi)|_V$ .

**Proof sketch:** (i) and (ii) follow from Lemma 4.

(iii) It suffices to show  $M|_V \in \mathcal{M}$  for each  $M \models \varphi$  where  $\mathcal{M}$  is defined as in Eq. (6). It is clear that  $M|_V \in \mathcal{M}$  if  $M \models \text{CIRC}[\varphi; P; Q]$ . In the case  $M \models \varphi$  and  $M \not\models \text{CIRC}[\varphi; P; Q]$ , there exists  $M' \models \text{CIRC}[\varphi; P; Q]$  such that  $M' <^{P; Q} M$ , which implies  $M'|_V <^{P; Q} M|_V$  due to  $V \subseteq F \cup Q$ . Therefore,  $M|_V \in \mathcal{M}$ . ■

By Lemma 5 (iii), we have the following representation theorem for  $[P; Q]$ -forgetting (under some condition).

**Proposition 9** *Let  $\varphi(P, Q)$  be a formula and  $V \subseteq F \cup Q$ . The following statements are equivalent:*

- (i)  $\text{F}_{[P; Q]}(\varphi, V)$ ,
- (ii)  $\text{F}(\varphi, V)$ ,
- (iii)  $\{\xi | \varphi \models \xi \text{ and } \text{Var}(\xi) \cap V = \emptyset\}$ .

Moreover, under the same condition,  $[P; Q]$ -forgetting possesses several other interesting properties of classical forgetting.

**Proposition 10** *Let  $\varphi(P, Q, F), \psi(P, Q, F)$  be two formulas,  $V, V_1$  and  $V_2$  are subsets of  $F \cup Q$ . Then*

- (i)  $\text{F}_{[P; Q]}(\varphi, V_1 \cup V_2) \equiv \text{F}_{[P; Q]}(\text{F}_{[P; Q]}(\varphi, V_1), V_2)$ .
- (ii)  $\text{F}_{[P; Q]}(\varphi \vee \psi, V) \equiv \text{F}_{[P; Q]}(\varphi, V) \vee \text{F}_{[P; Q]}(\psi, V)$ .
- (iii)  $\text{F}_{[P; Q]}(\varphi \wedge \psi, V) \equiv \text{F}_{[P; Q]}(\varphi, V) \wedge \psi$  if  $\text{IR}(\psi, V)$ .
- (iv)  $\text{F}_{[P; Q]}(\varphi, V_1) \models \text{F}_{[P; Q]}(\varphi, V_2)$  if  $V_1 \subseteq V_2$ .

The next theorem provides some sufficient (and necessary) conditions for “consequence preserving”.

**Theorem 4** *Let  $\varphi(P, Q)$  be a formula,  $V \subseteq \mathcal{A}$ ,  $\xi$  a formula with  $\text{IR}(\xi, V)$  and  $\psi \equiv \text{F}_{[P; Q]}(\varphi, V)$ .*

- (i) If  $\text{CIRC}[\varphi; P; Q] \models \xi$ , then  $\text{CIRC}[\psi; P; Q] \models \xi$ .
- (ii) Under the condition  $V \subseteq Q$  or  $P \subseteq V$ , it holds that  $\text{CIRC}[\varphi; P; Q] \models \xi$  iff  $\text{CIRC}[\psi; P; Q] \models \xi$ .
- (iii) Whenever  $\text{MM}_{[P; Q]}(\varphi)|_V$  is  $[P; Q]$ -minimal, it holds that  $\text{CIRC}[\varphi; P; Q] \models \xi$  iff  $\text{CIRC}[\psi; P; Q] \models \xi$ .

## Uniform interpolation

Following the notion of uniform interpolation for answer set programs (Gabbay, Pearce, and Valverde 2011), we say that a class of circumscription theories on  $\mathcal{L}^{\mathcal{A}}$  has the *uniform interpolation property* whenever, for any pair  $\langle \varphi(P, Q), \xi \rangle$  of formulas in the subclass, there is a formula  $\psi$  with  $\text{Var}(\psi) \subseteq \text{Var}(\varphi) \cap \text{Var}(\xi)$  such that

$$\text{CIRC}[\varphi; P; Q] \models \xi \text{ iff } \text{CIRC}[\psi; P; Q] \models \xi. \quad (7)$$

In this case,  $\psi$  is called the *uniform interpolation* of  $\langle \varphi, \xi \rangle$ .

We note that the notion of uniform interpolation property is slightly different from the ones in classical logics (Goranko and Otto 2007) and Amir’s interpolation property for circumscription (Amir 2002), in which it requires  $\varphi \models \psi$  and the condition (7) where  $\text{Var}(\psi) \subseteq \text{Var}(\varphi) \cap (\text{Var}(\xi) \cup P)$ .

The following two corollaries follow from Theorem 4.

**Corollary 5** *A subclass circumscription  $\text{CIRC}[\varphi; P; Q]$  of  $\mathcal{L}^{\mathcal{A}}$  has the uniform interpolation property iff  $\text{MM}_{[P; Q]}(\varphi)|_V$  is  $[P; Q]$ -minimal for each  $\text{CIRC}[\varphi; P; Q]$  in the subclass and  $V \subseteq \mathcal{A}$ .*

The next corollary shows a conditional uniform interpolation property for circumscription.

**Corollary 6** *If  $P \subseteq V$  or  $V \subseteq Q$  then  $\langle \text{CIRC}[\varphi; P; Q], \xi \rangle$  has a uniform interpolation where  $\text{IR}(\xi, V)$  holds.*

## An algorithm and computational complexities

The proof of Theorem 2 actually hints a brute force algorithm for computing  $[P; Q]$ -forgetting.

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### Algorithm 1: Computing $[P; Q]$ -forgetting result

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**Input** : A formula  $\varphi(P, Q)$  and  $V \subseteq \mathcal{A}$   
**Output**: A result of  $[P; Q]$ -forgetting  $V$  from  $\varphi$

```

1  $\mathcal{M} \leftarrow \text{Mod}(\text{CIRC}[\varphi; P; Q])|_V$ ;
2 foreach  $M$  s.t.  $M \models \varphi$  and  $M \notin \mathcal{M}$  do
3   | if  $\exists M' \in \mathcal{M}$  such that  $M' <^{P; Q} M|_V$  then
4   |   |  $\mathcal{M} \leftarrow \mathcal{M} \cup \{M|_V\}$ ;
5   | end
6 end
7  $\psi \leftarrow \bigvee_{M \in \mathcal{M}} (\bigwedge (M \cup \neg(\overline{V} \setminus M)))$ ;
8 return  $\psi$ ;
```

---

The algorithm first find out all models of the input circumscription theory and then remove all variables of  $V$  from

each of these models. From the resulting collection of interpretations, we choose those minimal ones, which are actually the models of the result of forgetting in the circumscription theory. Based on these minimal models, we are able to construct the result of  $[P; Q]$ -forgetting.

**Theorem 7** *Algorithm 1 computes  $F_{[P;Q]}(\varphi, V)$ .*

Since  $|\mathcal{M}|$  in Algorithm 1 is possibly exponential in the size of  $\varphi(P, Q)$  and  $V$ , it generates a formula which is in exponentially large in the worst case.

It is known that the *model checking problem* for propositional circumscription, i.e. deciding if a model  $M$  of a formula  $\varphi$  is a model of  $\text{CIRC}[\varphi; P; Q]$ , is coNP-complete, cf. Theorem 3 of (Cadoli 1992), whereas the *inference problem*, i.e., deciding if a clause is derivable from  $\text{CIRC}[\varphi; P; Q]$ , is  $\Pi_2^P$ -complete (Eiter and Gottlob 1993). Thus, we are able to show the following results of computational complexity for  $[P; Q]$ -forgetting.

**Proposition 11** *Let  $\varphi(P, Q)$  be a formula,  $V \subseteq \mathcal{A}$  and  $\xi$  a formula with  $\text{IR}(\xi, V)$ . The problem of deciding if  $\text{CIRC}[F_{[P;Q]}(\varphi, V); P; Q] \models \xi$  is in  $\Pi_3^P$ .*

**Proof sketch:** If  $\text{CIRC}[F_{[P;Q]}(\varphi, V); P; Q] \not\models \xi$  then there exists  $M \models \text{CIRC}[\varphi; P; Q]$  such that  $M \upharpoonright_V$  is  $[P; Q]$ -minimal wrt  $MM_{[P;Q]}(\varphi) \upharpoonright_V$ , which is tractable by calling an  $\Pi_2^P$ -oracle. As model checking for propositional circumscription is in coNP, the problem is in  $\Pi_3^P$ . ■

**Theorem 8** *Let  $\varphi(P, Q)$  be a formula,  $V \subseteq \mathcal{A}$ . Deciding if  $\varphi \equiv F_{[P;Q]}(\varphi, V)$  is  $\Pi_2^P$ -complete.*

**Proof sketch:** Its hardness follows from the fact  $\varphi \equiv F_{[P;Q]}(\varphi, V)$  iff  $\varphi \equiv F(\varphi, V)$ , which is  $\Pi_2^P$ -complete. ■

## Concluding Remarks

For a propositional circumscription  $\text{CIRC}[\varphi; P; Q]$ , we have defined the notions of strong  $[P; Q]$ -equivalence and  $[P; Q]$ -irrelevance, which lay a foundation for our theory of  $[P; Q]$ -forgetting for circumscription.  $[P; Q]$ -forgetting is a natural generalization of forgetting for propositional logic. We investigated various properties of  $[P; Q]$ -forgetting, and provided some preliminary results on computational complexity of  $[P; Q]$ -forgetting as well. Our work shows that the task of defining a suitable concept of forgetting for circumscription is non-trivial, which can be seen from both the definition of  $[P; Q]$ -forgetting and its properties that are different from classical forgetting and forgetting for answer set programming.

There are a few interesting issues for future work. First, given the complexity of  $[P; Q]$ -forgetting, it would infeasible to develop efficient algorithms. So, it is worthwhile to identify useful classes of circumscription theories for which  $[P; Q]$ -forgetting is tractable or has lower complexity. Another challenging issue is to extend our theory of  $[P; Q]$ -forgetting to first-order circumscription. Since the semantics of several DL ontology languages is defined in terms of circumscription (Bonatti, Lutz, and Wolter 2009), it is worthwhile to investigate applications of  $[P; Q]$ -forgetting in these ontology languages.

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