

## 3515ICT Theory of Computation

### Tutorial problems: Turing machines, decidability and undecidability

1. Give implementation level and formal descriptions of a Turing machine to recognise the language  $L = \{ a^n b^n c^n \mid n \geq 0 \}$ . You may assume either a single- or doubly-infinite tape.
2. Give implementation level and formal descriptions of a Turing machine to compute the sum of integers  $m$  and  $n$ . The input should have the form  $1^{m+1}01^{n+1}$ , with the head on the leftmost 1, and the output should have the form  $1^{m+n+1}$ , with the head again on the leftmost 1.
3. Give an implementation level description of a Turing machine to compute Ackermann's function, defined by  $A(0, n) = n + 1$ ,  $A(m + 1, 0) = A(m, 1)$ ,  $A(m + 1, n + 1) = A(m, A(m + 1, n))$ . The input and output should have the same form as in the previous question. (Difficult)
4. (Hopcroft *et al.*, Exercise 8.4.3) Give implementation level descriptions of nondeterministic Turing machines — possibly a multitape machines — that recognise the following languages. Try to exploit nondeterminism to avoid iteration and keep computations short and your Turing machines small.
  - (a) The set of all strings of  $a$ 's and  $b$ 's that contain two nonoverlapping occurrences of some substring of length 100, *i.e.*,
$$L = \{ wxyz \in \{a, b\}^* \mid |x| = 100 \}$$
  - (b) The language of all strings of the form  $w_1 \# w_2 \# \dots \# w_n$ , for any  $n \geq 0$ , such that each  $w_i \in \{0, 1\}^+$  and, for some  $j$ ,  $w_j$  is the binary representation of the integer  $j$ .
5. Prove that every context-free language is Turing-recognisable. That is, prove that every language accepted by some PDA is also accepted by some Turing machine.
6.
  - (a) Construct a 2-stack PDA to accept the language  $L_{abc} = \{ a^n b^n c^n \mid n \geq 0 \}$ .
  - (b) Construct a 2-stack PDA to accept the language  $L_{ww} = \{ ww \mid w \in \{a, b\}^* \}$ .
7. Prove that every Turing-machine (with a doubly-infinite tape) can be simulated by a 2-stack PDA.
8. Define a “queue machine”. Prove that if a language is accepted by some TM, and hence by some 2-stack PDA, then it is also accepted by some queue machine.

**Hint** There is a standard representation of a queue using two stacks with  $O(1)$  amortised time for *put* and *get* operations. For this exercise we need to be able to represent two stacks using a single queue.
9. Give a detailed encoding of arbitrary Turing machines in some fixed alphabet.
10. Give a detailed implementation level description of a universal Turing machine  $U$ . The machine  $U$  takes as input an encoding (previous question) of a Turing machine  $M$  and a string  $w$ . It should simulate the behaviour of  $M$  when started with  $w$  on its input tape, and *accept* (resp., *reject*) iff  $M$  applied to  $w$  accepts (resp., rejects).

**Hint.** Use a 3-tape machine. Store  $\langle M, w \rangle$  on the first tape, store the current tape of  $M$  on the second tape, and store the current state of  $M$  on the third tape.

11. (Sipser, Problem 4.12) Show that the language

$$A = \{ \langle R, S \rangle \mid R \text{ and } S \text{ are regular expressions and } L(R) \subseteq L(S) \}$$

is decidable.

12. (Sipser, Problem 4.13) Let  $\Sigma = \{0, 1\}$ . Show that the language

$$B = \{ \langle G \rangle \mid G \text{ is a CFG over } \Sigma \text{ and } 1^* \cap L(G) \neq \emptyset \}$$

is decidable.

13. (Sipser, Problem 4.14) Let  $\Sigma = \{0, 1\}$ . Show that the language

$$C = \{ \langle G \rangle \mid G \text{ is a CFG over } \Sigma \text{ and } 1^* \subseteq L(G) \}$$

is decidable.

14. (Sipser, Problem 4.24) Show that the language

$$PAL_{DFA} = \{ \langle M \rangle \mid M \text{ is a DFA that accepts some palindrome} \}$$

is decidable. **Hint.** Use theorems about CFLs.

15. (Sipser, Problem 5.9) Show that the language

$$T = \{ \langle M \rangle \mid M \text{ is a TM that accepts } w^R \text{ iff it accepts } w \}$$

is undecidable.

16. (Sipser, Problems 5.17) Show that the Post Correspondence Problem is decidable over the unary alphabet  $\Sigma = \{1\}$ .

17. (Sipser, 5.20) Prove that there exists an undecidable subset of  $\{1\}$ .

18. (Sipser, 5.29) Show that both conditions of Rice's Theorem are necessary for proving that the property  $P$  to be undecidable.

19. (Sipser, 5.33) Let

$$S = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \{ \langle M \rangle \} \}.$$

Show that neither  $S$  nor  $\bar{S}$  is Turing-recognisable.

20. (Hopcroft *et al.*, Exercise 9.3.4) We know by Rice's theorem that none of the following problems are decidable. However, are they Turing-recognisable or not?

- (a) Does  $L(M)$  contain at least two strings?
- (b) Is  $L(M)$  infinite?
- (c) Is  $L(M)$  a context-free language?
- (d) Is  $L(M) = (L(M))^R$ ?