1. (a) Prove by mathematical induction that, for all integers \( N \geq 0 \), \( N(N^2+5) \) is divisible by 6.

(b) Prove the same result without using mathematical induction.

2. Let \( F_n = 2^{2n} + 1 \), for \( n \geq 0 \). Prove by mathematical induction that

\[
F_n = \prod_{k=0}^{n-1} F_k + 2.
\]

(Here, \( \Pi \) denotes product; the product of an empty set is 1.)

3. Prove by mathematical induction that, for all integers \( N \geq 4 \), \( N! > 2^N \).

4. A binary tree is full if every internal node has exactly 2 subtrees. Prove that a full binary tree with \( n \) leaves has \( 2^n - 1 \) nodes by integer induction and by structural induction.

5. Define a \( k \)-ary tree to be full if every internal node has exactly \( k \) subtrees. Find a formula for the number of nodes in a full \( k \)-ary tree with \( n \) leaves, and prove it by structural induction.

6. Let \( w^R \) be the reverse of a string \( w \). Let \( w_1 \) and \( w_2 \) be strings in \( \{a, b\} \). Prove by mathematical induction on \( |w_1| \) that \( w_1^R w_2^R = (w_2 w_1)^R \).

7. Let \( w \) be a string in \( \{a, b\} \). Prove by mathematical induction on \( |w| \) that \( (w^R)^R = w \).

8. There are two ways to define “balanced parenthesis strings”:

(a) Grammatically (GB): The empty string \( \epsilon \) is balanced; if \( s \) is balanced then \( (s) \) is balanced; if \( s \) and \( t \) are balanced then \( st \) is balanced.

(b) By scanning (SB): \( s \) is balanced if and only if \( s \) has an equal number of left and right parentheses, and every prefix of \( s \) has at least as many left as right parentheses.

Prove that a string of parentheses is GB if and only if it is SB.

9. Prove by contradiction that the square root of 2 is not rational.

10. Prove that the following sets are countably infinite.

(a) The set of all strings in \( \{a, b\}^* \).

(b) The set of all positive rational numbers.

(c) The set of all full binary trees whose leaves are \( a \) or \( b \).